ON NEW APPROACH HADAMARD-TYPE INEQUALITIES FOR s-GEOMETRICALLY CONVEX FUNCTIONS

MEVLÜT TUNÇ□ AND İBRAHİM KARABAYIR△

Abstract. In this paper we achieve some new Hadamard type inequalities using elementary well known inequalities for functions whose first derivatives absolute values are s-geometrically and geometrically convex. And also we get some applications for special means for positive numbers.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions.

In recent years many authors established several inequalities connected to this fact. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard-type inequalities see [1]-[9].

In this section we will present definitions and some results used in this paper.

Definition 1. *Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$ is said to be convex if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 2. [1] *Let $s \in (0,1]$. A function $f : I \subset \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex in the second sense if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$.

It can be easily checked for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Recently, in [3], the concept of geometrically and s-geometrically convex functions was introduced as follows.

Date: November 15, 2012.
2000 *Mathematics Subject Classification.* 26A15, 26A51, 26D10.
Key words and phrases. s-geometrically convex, geometrically convex, Hadamard's inequality, Hölder's inequality, power mean inequality, means.
This paper is in final form and no version of it will be submitted for publication elsewhere.
□Corresponding Author.

Definition 3. [3] *A function $f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if*

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^t [f(y)]^{1-t} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 4. [3] *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a s-geometrically convex function if*

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s} \tag{1.4}$$

for some $s \in (0,1]$, where $x, y \in I$ and $t \in [0,1]$.

If $s = 1$, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

Example 1. [3] *Let $f(x) = x^s/s$, $x \in (0,1]$, $0 < s < 1$, $q \geq 1$, and then the function*

$$|f'(x)|^q = x^{(s-1)q} \tag{1.5}$$

is monotonically decreasing on $(0,1]$. For $t \in [0,1]$, we have

$$(s-1)\, q\, (t^s - t) \leq 0, \quad (s-1)\, q\, ((1-t)^s - (1-t)) \leq 0. \tag{1.6}$$

Hence, $|f'(x)|^q$ is s-geometrically convex on $(0,1]$ for $0 < s < 1$.

2. Hadamard's type inequalities

In order to prove our main theorems, we need the following lemma [2].

Lemma 1. [2] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$.If $f' \in L[a,b]$, then the following equality holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \\ = &\frac{b-a}{4}\left[\int_0^1 (-t)\, f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 t f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt\right]. \end{aligned} \tag{2.1}$$

A simple proof of this equality can be also done integrating by parts in the right hand side. The details are left to the interested reader.

The next theorems gives a new result of the upper Hermite-Hadamard inequality for s-geometrically convex functions.

In the following part of the paper;

$$\alpha(u,v) = |f'(a)|^u\, |f'(b)|^{-v}, \quad u, v \geq 0, \tag{2.2}$$

$$g_1(\alpha) = \begin{cases} \frac{1}{2} & \alpha = 1 \\ \frac{\alpha \ln \alpha - \alpha + 1}{(\ln \alpha)^2} & \alpha \neq 1 \end{cases} \tag{2.3}$$

and

$$g_2(\alpha) = \begin{cases} 1 & \alpha = 1 \\ \frac{\alpha - 1}{\ln \alpha} & \alpha \neq 1 \end{cases} \tag{2.4}$$

Theorem 1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable mapping on I°, $a, b \in I$ with $a < b$ and f' is integrable on $[a, b]$. If $|f'|$ is s-geometrically convex and monotonically decreasing on $[a, b]$, and $s \in (0, 1]$ then the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\ &\leq \frac{(b-a)}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left(g_1\left(\alpha\left(\frac{s}{2}, \frac{s}{2} \right) \right) + g_1\left(\alpha\left(\frac{-s}{2}, \frac{-s}{2} \right) \right) \right). \end{aligned} \tag{2.5}$$

Proof. Since $|f'|$ is a s-geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 1, we get

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\ \leq\ & \frac{b-a}{4} \left[\int_0^1 |-t| \left| f'\left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) \right| dt + \int_0^1 |t| \left| f'\left(\frac{1+t}{2} b + \frac{1-t}{2} a \right) \right| dt \right] \\ \leq\ & \frac{b-a}{4} \left[\int_0^1 |-t| \left| f'\left(a^{\frac{1+t}{2}} b^{\frac{1-t}{2}} \right) \right| dt + \int_0^1 |t| \left| f'\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}} \right) \right| dt \right] \\ \leq\ & \frac{b-a}{4} \left[\int_0^1 |-t| \left| f'(a) \right|^{\left(\frac{1+t}{2}\right)^s} \left| f'(b) \right|^{\left(\frac{1-t}{2}\right)^s} dt + \int_0^1 |t| \left| f'(b) \right|^{\left(\frac{1+t}{2}\right)^s} \left| f'(a) \right|^{\left(\frac{1-t}{2}\right)^s} dt \right] \end{aligned}$$

If $0 < k \leq 1$, $0 < m, n \leq 1$,

$$k^{m^n} \leq k^{mn} \tag{2.6}$$

When $|f'(a)| = |f'(b)| = 1$, by(2.6), we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{4}$$

When $0 < |f'(a)|, |f'(b)| < 1$, by(2.6), we get

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\ \leq\ & \frac{b-a}{4} \left[\int_0^1 |-t| \left| f'(a) \right|^{s\left(\frac{1+t}{2}\right)} \left| f'(b) \right|^{s\left(\frac{1-t}{2}\right)} dt + \int_0^1 |t| \left| f'(b) \right|^{s\left(\frac{1+t}{2}\right)} \left| f'(a) \right|^{s\left(\frac{1-t}{2}\right)} dt \right] \\ =\ & \frac{b-a}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left[\int_0^1 |-t| \left| \frac{f'(a)}{f'(b)} \right|^{\frac{st}{2}} dt + \int_0^1 |t| \left| \frac{f'(b)}{f'(a)} \right|^{\frac{st}{2}} dt \right] \\ =\ & \frac{(b-a)}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left(g_1\left(\alpha\left(\frac{s}{2}, \frac{s}{2} \right) \right) + g_1\left(\alpha\left(\frac{-s}{2}, \frac{-s}{2} \right) \right) \right) \end{aligned}$$

which completes the proof. □

Theorem 2. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$ and $f' \in L([a, b])$. If $|f'|^q$ is s-geometrically convex and monotonically decreasing*

on $[a,b]$ *for* $p,q>1$ *and* $s\in(0,1]$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left\{\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right\},
\end{aligned}\tag{2.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f'|^q$ is a s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1 and the well known Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1|-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1|t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4}\left\{\left[\int_0^1 t^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 t^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left\{\left[\int_0^1\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}+\left[\int_0^1\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left\{\left[\int_0^1\left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}\right)^q dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1\left(|f'(b)|^{\left(\frac{1+t}{2}\right)^s}|f'(a)|^{\left(\frac{1-t}{2}\right)^s}\right)^q dt\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

When $|f'(a)|=|f'(b)|=1$, by(2.6), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}$$

When $0<|f'(a)|,|f'(b)|<1$, by(2.6), we get

$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left\{\left[\int_0^1 |f'(a)|^{sq\left(\frac{1+t}{2}\right)}|f'(b)|^{sq\left(\frac{1-t}{2}\right)}dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 |f'(b)|^{sq\left(\frac{1+t}{2}\right)}|f'(a)|^{sq\left(\frac{1-t}{2}\right)}dt\right]^{\frac{1}{q}}\right\}$$
$$= \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}|f'(a)f'(b)|^{\frac{s}{2}}\left\{\left(\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|\frac{f'(b)}{f'(a)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}\right\}$$
$$= \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}|f'(a)f'(b)|^{\frac{s}{2}}\left\{\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right\}$$

which completes the proof. □

Corollary 1. *Let $f : I \subseteq (0,\infty) \to (0,\infty)$ be differentiable on I°, $a,b \in I$ with $a < b$, and $f' \in L([a,b])$. If $|f'|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $p,q > 1$ and $s \in (0,1]$, then*

i) When $p = q = 2$, one has

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{4\sqrt{3}}|f'(a)f'(b)|^{\frac{s}{2}}\left\{\sqrt{g_2(\alpha(s,s))}+\sqrt{g_2(\alpha(-s,-s))}\right\}$$

ii) If we take $s = 1$ in (2.7), we have for geometrically convex, one has

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}|f'(a)f'(b)|^{\frac{1}{2}}\left\{[g_2(\alpha(q,q))]^{\frac{1}{q}}+[g_2(\alpha(-q,-q))]^{\frac{1}{q}}\right\}$$

Theorem 3. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a,b \in I$ with $a < b$ and $f' \in L([a,b])$. If $|f'|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, for $q \geq 1$ and $s \in (0,1]$, then*

(2.8)
$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left|\frac{f'(a)}{f'(b)}\right|^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left|\frac{f'(b)}{f'(a)}\right|^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right\}.$$

Proof. Since $|f'|^q$ is a s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1 and the well known power mean integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4}\left\{\left[\int_0^1 tdt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 tdt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right]^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\int_0^1 t\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}+\left[\int_0^1 \left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\int_0^1 t\left|f'(a)^{q\left(\frac{1+t}{2}\right)^s}f'(b)^{q\left(\frac{1-t}{2}\right)^s}\right|dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 t\left|f'(b)^{q\left(\frac{1+t}{2}\right)^s}f'(a)^{q\left(\frac{1-t}{2}\right)^s}\right|dt\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

When $|f'(a)|=|f'(b)|=1$, by(2.6), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4}$$

When $0<|f'(a)|,|f'(b)|<1$, by(2.6), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\int_0^1 t\left|f'(a)^{sq\left(\frac{1+t}{2}\right)}f'(b)^{sq\left(\frac{1-t}{2}\right)}\right|dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 t\left|f'(b)^{sq\left(\frac{1+t}{2}\right)}f'(a)^{sq\left(\frac{1-t}{2}\right)}\right|dt\right]^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}}\int_0^1 t\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\left[\left|\frac{f'(b)}{f'(a)}\right|^{\frac{sq}{2}}\int_0^1 t\left|\frac{f'(b)}{f'(a)}\right|^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left|\frac{f'(a)}{f'(b)}\right|^{\frac{s}{2}}\left[\int_0^1 t\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\left|\frac{f'(b)}{f'(a)}\right|^{\frac{s}{2}}\left[\int_0^1 t\left|\frac{f'(b)}{f'(a)}\right|^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left|\frac{f'(a)}{f'(b)}\right|^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left|\frac{f'(b)}{f'(a)}\right|^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof. □

Theorem 4. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$ and $f' \in L([a, b])$. If $|f'|$ is s-geometrically convex and monotonically decreasing on $[a, b]$ for $\mu_1, \mu_2, \eta_1, \eta_2 > 0$ with $\mu_1 + \eta_1 = 1$ and $\mu_2 + \eta_2 = 1$ and $s \in (0, 1]$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\;& \frac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left\{\frac{(1+\mu_2)\mu_1^2+(1+\mu_1)\mu_2^2}{(1+\mu_1)(1+\mu_2)}\right. \\ &\left.+\eta_1 g_2\left(\alpha\left(\frac{s}{2\eta_1},\frac{s}{2\eta_1}\right)\right)+\eta_2 g_2\left(\alpha\left(\frac{s}{2\eta_2},\frac{s}{2\eta_2}\right)\right)\right\}. \end{aligned} \tag{2.9}$$

Proof. Since $|f'|$ is a s-geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 1, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\;& \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right] \\ \leq\;& \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|dt+\int_0^1 |t|\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|dt\right] \\ \leq\;& \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{\left(\frac{1-t}{2}\right)^s}dt+\int_0^1 |t|\left|f'(b)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{\left(\frac{1-t}{2}\right)^s}dt\right] \end{aligned}$$

When $0 < |f'(a)|, |f'(b)| \leq 1$, by(2.6), we get

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\;& \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'(a)\right|^{s\left(\frac{1+t}{2}\right)}\left|f'(b)\right|^{s\left(\frac{1-t}{2}\right)}dt+\int_0^1 |t|\left|f'(b)\right|^{s\left(\frac{1+t}{2}\right)}\left|f'(a)\right|^{s\left(\frac{1-t}{2}\right)}dt\right] \\ =\;& \frac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left[\int_0^1 |-t|\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2}}dt+\int_0^1 |t|\left|\frac{f'(b)}{f'(a)}\right|^{\frac{st}{2}}dt\right] \end{aligned} \tag{2.10}$$

for all $t \in [0,1]$. Using the well known inequality $mn \leq \mu m^{\frac{1}{\mu}} + \eta n^{\frac{1}{\eta}}$, on the right side of (2.10), we get

$$\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\
\leq\ & \frac{b-a}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left\{ |-t| \left| \frac{f'(a)}{f'(b)} \right|^{\frac{st}{2}} dt + \int_0^1 |t| \left| \frac{f'(b)}{f'(a)} \right|^{\frac{st}{2}} dt \right\} \\
\leq\ & \frac{b-a}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left\{ \mu_1 \int_0^1 |-t|^{\frac{1}{\mu_1}} dt + \eta_1 \int_0^1 \left| \frac{f'(a)}{f'(b)} \right|^{\frac{s}{2\eta_1} t} dt \right. \\
& \left. + \mu_2 \int_0^1 |t|^{\frac{1}{\mu_2}} dt + \eta_2 \int_0^1 \left| \frac{f'(b)}{f'(a)} \right|^{\frac{s}{2\eta_2} t} dt \right\} \\
=\ & \frac{b-a}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left[\frac{\mu_1^2}{1+\mu_1} + \eta_1 \int_0^1 \left| \frac{f'(a)}{f'(b)} \right|^{\frac{s}{2\eta_1} t} dt + \frac{\mu_2^2}{1+\mu_2} + \eta_2 \int_0^1 \left| \frac{f'(b)}{f'(a)} \right|^{\frac{s}{2\eta_2} t} dt \right] \\
=\ & \frac{b-a}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left\{ \frac{\mu_1^2}{1+\mu_1} + \eta_1 g_2 \left(\alpha \left(\frac{s}{2\eta_1}, \frac{s}{2\eta_1} \right) \right) \right. \\
& \left. + \frac{\mu_2^2}{1+\mu_2} + \eta_2 g_2 \left(\alpha \left(\frac{s}{2\eta_2}, \frac{s}{2\eta_2} \right) \right) \right\} \\
=\ & \frac{b-a}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left\{ \frac{(1+\mu_2)\mu_1^2 + (1+\mu_1)\mu_2^2}{(1+\mu_1)(1+\mu_2)} \right. \\
& \left. + \eta_1 g_2 \left(\alpha \left(\frac{s}{2\eta_1}, \frac{s}{2\eta_1} \right) \right) + \eta_2 g_2 \left(\alpha \left(\frac{s}{2\eta_2}, \frac{s}{2\eta_2} \right) \right) \right\}
\end{aligned}$$

and we get, in here, if $|f'(a)| = |f'(b)| = 1$, we get

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{4} \left[\frac{(1+\mu_2)\mu_1^2 + (1+\mu_1)\mu_2^2}{(1+\mu_1)(1+\mu_2)} + \eta_1 + \eta_2 \right]$$

which the proof is completed. □

3. Applications to special means for positive numbers

Let

$$\begin{aligned}
A(a,b) &= \frac{a+b}{2},\ L(a,b) = \frac{b-a}{\ln b - \ln a} \qquad (a \neq b), \\
L_p(a,b) &= \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p},\ a \neq b,\ p \in \mathbb{R},\ p \neq -1, 0
\end{aligned}$$

be the arithmetic, logarithmic, generalized logarithmic means for $a, b > 0$ respectively.

In the following propositions, $\alpha(u,v) = \frac{|f'(a)|^u}{|f'(b)|^v} = \frac{|a^{s-1}|^u}{|b^{s-1}|^v}$.

Proposition 1. *Let $0<a<b\leq 1$, with $a\neq b$, and $0<s<1$. Then, we have*

$$\begin{aligned} & \frac{1}{s}\left|A\left(a^{s},b^{s}\right)-L_{s}\left(a,b\right)^{s}\right| \\ \leq\ & \frac{(b-a)}{4}\left(ab\right)^{\frac{s}{2}(s-1)}\left\{\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\right)^{2}}\right. \\ & \left.+\frac{\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\right)^{2}}\right\} \end{aligned}$$

Proof. Let $f(x)=\frac{x^{s}}{s}$, $x\in(0,1]$, $0<s<1$, then $|f'(x)|=x^{s-1}$, $x\in(0,1]$ is a s-geometrically convex mapping. The assertion follows from Theorem 1 applied to s-geometrically convex mapping $|f'(x)|=x^{s-1}$, $x\in(0,1]$. □

Example 2. *Let $f(x)=\frac{x^{s}}{s}$, $x\in(0,1]$, $0<s<1$, then $|f'(x)|=x^{s-1}$, $x\in(0,1]$ is a s-geometrically convex mapping. If we apply in Theorem 1, for $s=0.5, a=0.89, b=0.9$, we get*

$$\begin{aligned} & \frac{1}{s}\left|\frac{a^{s}+b^{s}}{2}-\left(\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right)\right| \\ =\ & 4.921\,067\,116\times 10^{-6} \\ \leq\ & \frac{(b-a)}{4}\left|ab\right|^{\frac{s}{2}(s-1)} \\ & \times\left(\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\right)^{2}}+\frac{\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\right)^{2}}\right) \\ =\ & 2.570\,313\,847\times 10^{-3} \end{aligned}$$

And similarly, if we apply for $s=0.2, a=0.15, b=0.6$, we obtain

$$9.780\,804\,473\times 10^{-2}\leq 0.136\,819\,309\,576\,863\,680\,170\,486$$

for $s=0.75, a=0.45, b=0.86$, we obtain

$$6.115\,413\,651\times 10^{-2}\leq 0.112\,144\,032\,368\,736\,206\,184\,243$$

etc.

Proposition 2. *Let $0<a<b\leq 1$,with $a\neq b$, and $0<s<1$, and $p,q>1$. Then, we have*

$$\begin{aligned} & \frac{1}{s}\left|A\left(a^{s},b^{s}\right)-L_{s}\left(a,b\right)^{s}\right| \\ \leq\ & \frac{(b-a)}{4\left(p+1\right)^{\frac{1}{p}}}\left|ab\right|^{\frac{s}{2}(s-1)}\left(b^{s(1-s)/2}+a^{s(1-s)/2}\right)\left[L\left(a^{(s-1)\frac{sq}{2}},b^{(s-1)\frac{sq}{2}}\right)\right]^{\frac{1}{q}}. \end{aligned}$$

Proof. The assertion follows from Theorem 2 applied to s-geometrically convex mapping $|f'(x)|=\frac{x^{s}}{s}$, $x\in(0,1]$. □

Proposition 3. *Let $0 < a < b \leq 1$, with $a \neq b$, and $0 < s < 1$, and $q \geq 1$. Then, we have*

$$\begin{aligned}
&\frac{1}{s}\left|A\left(a^{s},b^{s}\right)-L_{s}\left(a,b\right)^{s}\right| \\
\leq\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left|\frac{a}{b}\right|^{\frac{s}{2}(s-1)}\left[\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}\ln\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}-\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}\right)^{2}}\right]^{\frac{1}{q}}\right. \\
&\left.+\left|\frac{b}{a}\right|^{\frac{s}{2}(s-1)}\left[\frac{\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}+\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}-1}{\left(\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}\right)^{2}}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. The assertion follows from Theorem 3 applied to s-geometrically convex mapping $\left|f'\left(x\right)\right| = \frac{x^{s}}{s}$, $x \in (0,1]$. □

Proposition 4. *Let $0 < a < b \leq 1$, with $a \neq b$, and $0 < s < 1$, and $q \geq 1$. Then, we have*

$$\begin{aligned}
&\frac{1}{s}\left|A\left(a^{s},b^{s}\right)-L_{s}\left(a,b\right)^{s}\right| \\
\leq\ & \frac{b-a}{4}\left|ab\right|^{\frac{s}{2}(s-1)} \\
&\times\left[\frac{\left(1+\mu_{2}\right)\mu_{1}^{2}+\left(1+\mu_{1}\right)\mu_{2}^{2}}{\left(1+\mu_{1}\right)\left(1+\mu_{2}\right)}+\eta_{1}\left[\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_{1}}}-1}{\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_{1}}}}\right]+\eta_{2}\left[\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_{2}}}-1}{\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_{2}}}}\right]\right].
\end{aligned}$$

Proof. The assertion follows from Theorem 4 applied to s-geometrically convex mapping $\left|f'\left(x\right)\right| = \frac{x^{s}}{s}$, $x \in (0,1]$. □

References

[1] H. Hudzik and L. Maligranda: Some remarks on s-convex functions, Aequationes Math., Vol. 48 (1994), 100–111.

[2] M. Alomari, M. Darus, U. S. Kırmacı: Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Comp. and Math. with Appl., Vol.59 (2010), 225-232.

[3] T.-Y. Zhang, A.-P. Ji and F. Qi: On Integral ınequalities of Hermite-Hadamard Type for s-Geometrically Convex Functions. Abstract and Applied Analysis. doi:10.1155/2012/560586.

[4] S.S. Dragomir, R.P. Agarwal: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl Math Lett, Vol. 11 No:5, (1998) 91–95.

[5] J. Hadamard: Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58, (1893) 171–215.

[6] M. Tunç: On some new inequalities for convex fonctions, Turk. J. Math. 36 (2012), 245-251.

[7] D. S. Mitrinović, J. Pečarić and A. M. Fink: Classical and new inequalities in analysis, KluwerAcademic, Dordrecht, 1993.

[8] S. S. Dragomir and C. E. M. Pearce: Selected topics on Hermite-Hadamard inequalities and applications, RGMIA monographs, Victoria University, 2000. [Online: http://www.staff.vu.edu.au/RGMIA/monographs/hermite-hadamard.html].

[9] J. E. Pečarić, F. Proschan and Y. L. Tong: Convex Functions, Partial Orderings, and Statistical Applications, Academic Press Inc., 1992.

$^{\square}$Department of Mathematics, Faculty of Science and Arts, Kilis 7 Aralik University, Kilis, 79000, Turkey.
E-mail address: $^{\square}$mevluttunc@kilis.edu.tr

$^{\triangle}$The Institute for Graduate Studies in Sciences and Engineering, Kilis 7 Aralik University, Kilis, 79000, Turkey.
E-mail address: $^{\triangle}$ikarabayir@kilis.edu.tr